Graph Theory in Physics: From Kirchhoff's Laws to Feynman Diagrams

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Abstract

Graph theory has found numerous applications in physics, from its early use in solving puzzle problems to its essential role in circuit analysis, modeling state transitions, and representing quantum interactions among fields and particles. This poster presents key insights from the forthcoming Springer publication *"Lectures on Graph Theory: Insights into Feynman Diagrams."* It highlights how graph-theoretical methods, alongside linear algebra, provide a unified framework for understanding Feynman diagrams, with applications to quantum field theory. The poster also explores classical applications such as Kirchhoff's laws, as well as their modern implications in fundamental physics.

Graph?

Mathematically: a collection of vertices and lines with **their incidence relation**.

Several Variants:

> *Most graph theorists use personalized terminology in their books, papers, and lectures. (Harary, F. "Graph Theory")*

For instance:

- Undirected graph
  - with/without self-loop
  - with/without parallel line
- Directed graph
  - with/without self-loop
  - with/without parallel line

Brief History

- Euler – the father of graph theory (1736 Königsberg Bridge Problem)

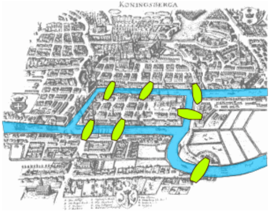


Figure: The Seven Bridges of Königsberg. (Wikipedia) Can you traverse all seven bridges exactly once?

- Kirchhoff – the theory of trees (1847)
- Cayley – chemical problem of trees (1874)
- Veblen – simplicial complex (1922)
- Feynman – Feynman diagram (1949)

Examples of Graphs in Physics

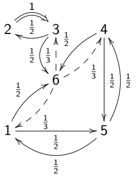
- Work-Energy Theorem: for a system, the change in kinetic energy KE is the total work W done on the system.

$$KE_{initial} \overset{\Downarrow W}{\dashrightarrow} KE_{final}$$

- The first law of Thermodynamics: the change in internal energy of a heat engine is the heat flow Q into the system minus the work done by the engine.

$$U_{initial} \overset{\Downarrow Q}{\underset{\Downarrow W}{\dashrightarrow}} U_{final}$$

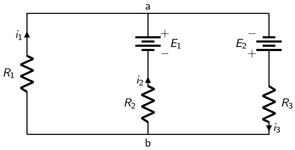
- Markov Chain: sequence of possible events with probability.



- Electrical Network: interconnection of electrical elements.

Kirchhoff's Rules

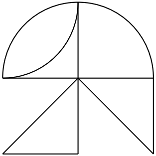
- Kirchhoff Junction Rule: vertex-by-vertex current conservation.
- Kirchhoff Loop Rule: loop-by-loop voltage conservation.



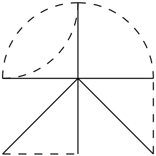
Question: How do we find and count the relevant loops?

Tree, Tree, Tree!

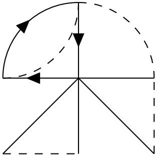
Removing lines from a given graph, we will eventually obtain an acyclic subgraph – a subgraph with no loops:



contains a spanning tree:

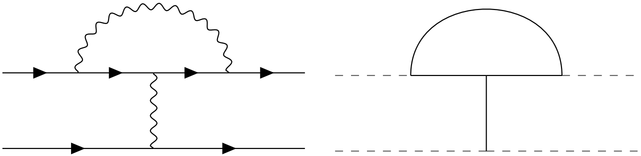


A maximal acyclic subgraph is called a spanning tree. By recovering a deleted line into a spanning tree, we can build a loop associated with a spanning tree.



Feynman Diagrams?

Objects in quantum field theory to visualize quantum interaction among fields and particles.
Graph-theoretically: graphs with *external legs*:



The external legs (dashed lines) represent interactions with the outside world, like ports in an electric circuit.

Kirchoff's Second Law for Feynman Diagrams

Similar to electric current flow, lines in a Feynman diagram carry flow – 4-momentum flow. In an electric circuit, the inner product of currents $\mathcal{I}$ and voltage $\mathcal{V}$ across each line is the electric power $\mathcal{I}^\top \mathcal{V}$. For an electric circuit with no ports, as the energy conservation, the electric power is conserved $\mathcal{I}^\top \mathcal{V} = 0$.
The graph-theoretical counterpart is known as Tellegen's theorem:

> *Among the theorems of network theory Tellegen's theorem is unusual in that it depends solely upon Kirchhoff's laws and the topology of the network. (Penfield, P. Spence, R. and Duinker, S. "Tellegen's Theorem and Electrical Networks")*

Question: Is there any voltage-like quantity associated with a Feynman diagram?
If such a quantity exists, thanks to Tellegen's theorem, we may obtain yet another conservation law!

To be published!

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Lectures on Graph Theory

Insights into Feynman Diagrams

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